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# Synchronization induced by common colored noise in limit cycle and chaotic systems

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We study the synchronization of dynamical systems induced by a common external noise which has an exponentially decaying time correlation. The synchronization threshold in the noise amplitude is discussed. We clarify how the synchronization threshold depends on the correlation decay rate of the noise for limit cycle and chaotic systems. The dependences are shown to be completely different: The threshold is independent of the correlation decay rate in limit cycle systems while it diverges in the limit of the large correlation decay rate and has the nonzero minimum value at a finite value of the correlation decay rate in chaotic systems. Effects of parameter mismatch between the driven systems on the synchronization quality are also discussed.

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## I. INTRODUCTION

In a variety of dynamical systems, a common external noise input to two independent and identical systems could give rise to synchronized motion of the two systems. We call this phenomenon *common-noise-induced synchronization* (CNIS). Synchronization is a key mechanism for the emergence of order and coherence in physical systems consisting of many dynamical elements. It is necessary for us to understand the conditions for synchronization under the application of common driving signals of various types including random noiselike signals.

Experimental evidence for the CNIS has been found for several systems as diverse as lasers, neuronal networks, and ecological systems. The CNIS has been demonstrated in a Nd:YAG microchip laser system, where the reproducibility of output from the laser was observed for repeated applications of the same input signal [1]. The reproducibility of spike timing in neurons has also been observed when the neurons were repeatedly driven by the same fluctuating input current [2]. In ecological systems, different populations separated by a large distance can synchronize with each other due to common environmental fluctuations such as climate changes [3]. On the other hand, there is also analytical and numerical evidence. It was analytically shown that the CNIS can occur in a wide class of limit cycle oscillators [4,5]. The CNIS was numerically demonstrated in chaotic maps [6] and in chaotic differential equations [7].

The above works have well established that the CNIS is a quite general phenomenon observable in a variety of different dynamical systems including both chaotic and limit cycle systems. Except for a few, most previous works assume the white noise case. Effects of the noise color have been discussed for a stable fixed point system [1] and limit cycle oscillators [5]. However, effects of the noise color have not been fully understood, especially in the case of chaotic systems. In the real world, due to memory effects, time correlation of noise is ubiquitous in nature and technology. Therefore, it is of great importance to understand the characteristics of CNIS induced by colored noise for a variety of dynamical systems including chaotic one. In this paper, we study the CNIS induced by colored noise, which has an exponentially decaying time correlation, for limit cycle and chaotic systems. Our analysis could directly apply to synchronization in electrical and optical systems, where the temporal correlations of driving signals can be controlled. We focus on how the synchronization threshold in the noise amplitude depends on the noise correlation decay rate and show that the threshold exhibits completely different dependences in chaotic and limit cycle systems. Effect of the time correlation is observed only in chaotic systems while the threshold is independent of the noise correlation decay rate in limit cycle systems. In the case of chaotic systems, the threshold diverges in the limit of large correlation decay rate and has the nonzero minimum value at a finite value of the correlation decay rate. We show that these dependences are universal characteristics of the CNIS in chaotic systems and limit cycle systems, respectively. The CNIS in the case of parameter mismatch between the two systems are also studied.

The present paper is organized as follows. In Sec. II, we describe the system configuration and the definition of the CNIS. In Sec. III, the CNIS in limit cycle oscillators is discussed. In Sec. IV, the CNIS in chaotic systems is discussed. In Sec. V, effects of parameter mismatch between the two driven systems is studied. Conclusions are drawn in Sec. VI.

## II. SYSTEM CONFIGURATION AND THE CNIS

Our investigation is of the noise-driven dynamical systems illustrated in Fig. 1. The two dynamical systems $S_1$ and $S_2$ are supposed to be identical and have different initial conditions. We consider the two cases that $S_1$ and $S_2$ are both limit cycle oscillators and that they are both chaotic systems.

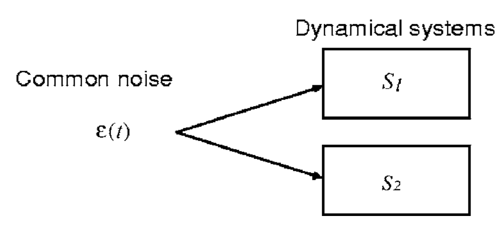


FIG. 1. Configuration of noise-driven systems.

 

These two systems are driven by a common Ornstein-Uhlenbeck (OU) noise input $\varepsilon(t)$, which is described by the equation

$$\dot{\varepsilon}(t) = -\gamma\varepsilon(t) + \gamma\eta(t), \quad (1)$$

where $\varepsilon \in \mathbf{R}$, $\gamma$ is a positive constant, and $\eta(t)$ represents the Gaussian white noise. The noise $\eta$ has the properties $\langle \eta(t) \rangle = 0$ and $\langle \eta(t)\eta(s) \rangle = 2D\delta(t-s)$, where $D$ is a positive constant, $\delta$ is Dirac's delta function, and $\langle \cdots \rangle$ denotes averaging over the realizations of $\eta$. The noise $\varepsilon$, which is driven by $\eta$, is an exponentially correlated noise with the properties

$$\langle \varepsilon(t) \rangle = 0, \quad (2)$$

$$\langle \varepsilon(t)\varepsilon(s) \rangle = D\gamma \exp[-\gamma|t-s|]. \quad (3)$$

We define the amplitude $\sigma$ of noise $\varepsilon$ by its standard deviation. From Eq. (3), $\sigma$ is given by $\sigma = \sqrt{D\gamma}$. The parameter $\gamma$ is corresponds to the inverse of noise correlation time. We call $\gamma$ the noise correlation decay rate.

Let $\boldsymbol{x}_1 \in \mathbf{R}^n$ and $\boldsymbol{x}_2 \in \mathbf{R}^n$ be the state variables of the systems $S_1$ and $S_2$, respectively. Consider two trajectories $\boldsymbol{x}_1(t)$ and $\boldsymbol{x}_2(t)$ with different initial conditions for the same realization of the noise $\varepsilon(t)$. The CNIS is said to occur if $\lim_{t\to\infty} \|\boldsymbol{x}_2(t) - \boldsymbol{x}_1(t)\| = 0$ for any initial conditions, where $\|\cdots\|$ represents the Euclidean norm. The linear stability of the complete synchronization state $\boldsymbol{x}_1 = \boldsymbol{x}_2$ is characterized by a quantity called the *largest conditional Lyapunov exponent* (LCLE), which will be defined later. A precise and useful criterion for the CNIS can be obtained in terms of the LCLE: i.e., the CNIS occurs if the LCLE is negative.

## III. CNIS IN LIMIT CYCLE SYSTEMS

First, we consider the case of limit cycle oscillators. We briefly describe the theory of Ref. [4] since our following argument is based on it. Let $\boldsymbol{X} \in \mathbf{R}^N$ be a state variable vector and consider the equation

$$\dot{\boldsymbol{X}} = \boldsymbol{F}(\boldsymbol{X}) + \boldsymbol{C}\eta(t), \quad (4)$$

where $\boldsymbol{F}$ is a smooth function, $\boldsymbol{C} \in \mathbf{R}^N$ is a constant vector, and $\eta(t)$ is the Gaussian white noise with the properties $\langle \eta(t) \rangle = 0$ and $\langle \eta(t)\eta(s) \rangle = 2D\delta(t-s), (D>0)$. A more general form of the noise term is considered in [4]. The noise-free system $\dot{\boldsymbol{X}} = \boldsymbol{F}(\boldsymbol{X})$ is assumed to have a limit cycle.

Consider two solutions $\boldsymbol{X}_1(t)$ and $\boldsymbol{X}_2(t)$ of Eq. (4) for the same realization of the noise $\eta(t)$: i.e.,

$$\boldsymbol{X}_1 = \boldsymbol{F}(\boldsymbol{X}_1) + \boldsymbol{C}\eta(t), \quad (5)$$

$$\boldsymbol{X}_2 = \boldsymbol{F}(\boldsymbol{X}_2) + \boldsymbol{C}\eta(t). \quad (6)$$

Suppose that $\boldsymbol{X}_1(t)$ and $\boldsymbol{X}_2(t)$ have an infinitesimally small difference at the initial time. Let $\delta\boldsymbol{X}$ be defined by $\delta\boldsymbol{X} = \boldsymbol{X}_2 - \boldsymbol{X}_1$, which represents the small deviation from the synchronization state. If we subtract Eq. (5) from Eq. (6), we have

$$\delta\dot{\boldsymbol{X}} = \boldsymbol{F}(\boldsymbol{X}_1 + \delta\boldsymbol{X}) - \boldsymbol{F}(\boldsymbol{X}_1). \quad (7)$$

If we expand Eq. (7) with respect to $\delta\boldsymbol{X}$ up to the first order, we have

$$\delta\dot{\boldsymbol{X}} = D\boldsymbol{F}(\boldsymbol{X}(t)) \cdot \delta\boldsymbol{X}, \quad (8)$$

where $D\boldsymbol{F}$ is the Jacobian matrix of $\boldsymbol{F}$ and $\boldsymbol{X}(t) = \boldsymbol{X}_1(t)$. The time evolution of $\delta\boldsymbol{X}$ is governed by the linearized equation (8). The LCLE is defined by

$$\lambda_g = \lim_{t\to\infty} \frac{1}{t} \ln \frac{\|\delta\boldsymbol{X}(t)\|}{\|\delta\boldsymbol{X}(0)\|}. \quad (9)$$

Equation (8) does not explicitly depend on $\eta$ and is of the same form for both the noise-free ($\eta=0$) and the noisy ($\eta \neq 0$) cases. However, the presence of noise $\eta$ in Eq. (4) affects the time evolution of $\boldsymbol{X}(t)$. The influence on $\delta\boldsymbol{X}(t)$ of the noise appears through $\boldsymbol{X}(t)$ in Eq. (8). Therefore, we note that there is a noise dependence of $\lambda_g$.

Equation (4) can be reduced to the one-dimensional equation of motion for the phase variable by the phase reduction method when $D$ is small. Based on the reduced equation of motion, it has been analytically shown that $\lambda_g$ is always negative for any $D>0$ in small $D$ region [4]. This result implies that the CNIS is caused by an arbitrarily small Gaussian white noise in any limit cycle system having the smooth function $\boldsymbol{F}$.

Let us consider the limit cycle oscillator driven by the OU noise, which is described by the equation

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{c}\varepsilon(t), \quad (10)$$

where $\boldsymbol{x} \in \mathbf{R}^n$, $\boldsymbol{f}$ is a smooth function, $\boldsymbol{c} \in \mathbf{R}^n$ is a constant vector, and $\varepsilon(t)$ is the OU noise defined by Eq. (1). The noise-free system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ is assumed to have a limit cycle. The LCLE in this case is defined by

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \ln \frac{\|\delta\boldsymbol{x}(t)\|}{\|\delta\boldsymbol{x}(0)\|}, \quad (11)$$

where $\delta\boldsymbol{x}(t)$ is a solution of the linearized equation

$$\delta\dot{\boldsymbol{x}} = D\boldsymbol{f}(\boldsymbol{x}(t)) \cdot \delta\boldsymbol{x}, \quad (12)$$

where $D\boldsymbol{f}$ is the Jacobian matrix of $\boldsymbol{f}$. Equation (12) is obtained in the same manner as Eq. (8).

The set of Eqs. (10) and (1) can be written in the form of Eq. (4) if we make the replacements $\boldsymbol{X} = (\boldsymbol{x}, \varepsilon)^T$, $\boldsymbol{F}(\boldsymbol{X}) = [\boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{c}\varepsilon, -\gamma\varepsilon]^T$, and $\boldsymbol{C} = (\boldsymbol{0}, \gamma)^T$, where $T$ represents the transposed vector. This set of equations also has a limit cycle in the case of $D=0$ because $\lim_{t\to\infty}\varepsilon(t) = 0$. The linearized equation corresponding to Eq. (8) is obtained as

$$\frac{d}{dt}\begin{pmatrix} \delta\boldsymbol{x} \\ \delta\varepsilon \end{pmatrix} = \begin{pmatrix} D\boldsymbol{f}(\boldsymbol{x}(t)) & \boldsymbol{c} \\ 0 & -\gamma \end{pmatrix} \begin{pmatrix} \delta\boldsymbol{x} \\ \delta\varepsilon \end{pmatrix}, \quad (13)$$

where $(\delta\boldsymbol{x}, \delta\varepsilon)^T$ corresponds to $\delta\boldsymbol{X}$. According to the theory in [4], we have

$$\lambda_g = \lim_{t\to\infty} \frac{1}{t} \ln \frac{\{[\delta\boldsymbol{x}(t)]^2 + [\delta\varepsilon(t)]^2\}^{1/2}}{\{[\delta\boldsymbol{x}(0)]^2 + [\delta\varepsilon(0)]^2\}^{1/2}} < 0 \quad (14)$$

for any initial variation $[\delta\boldsymbol{x}(0), \delta\varepsilon(0)]^T$ when $D>0$. Note that $\delta\varepsilon(0) = 0$ in our case because the two driven systems are subjected to the same realization of $\varepsilon(t)$. In the case of $\delta\varepsilon(0) = 0$, Eq. (13) reduces to Eq. (12) and $\lambda_g$ is the same as $\lambda$ defined by Eq. (11). Therefore, $\lambda = \lambda_g < 0$ holds for any

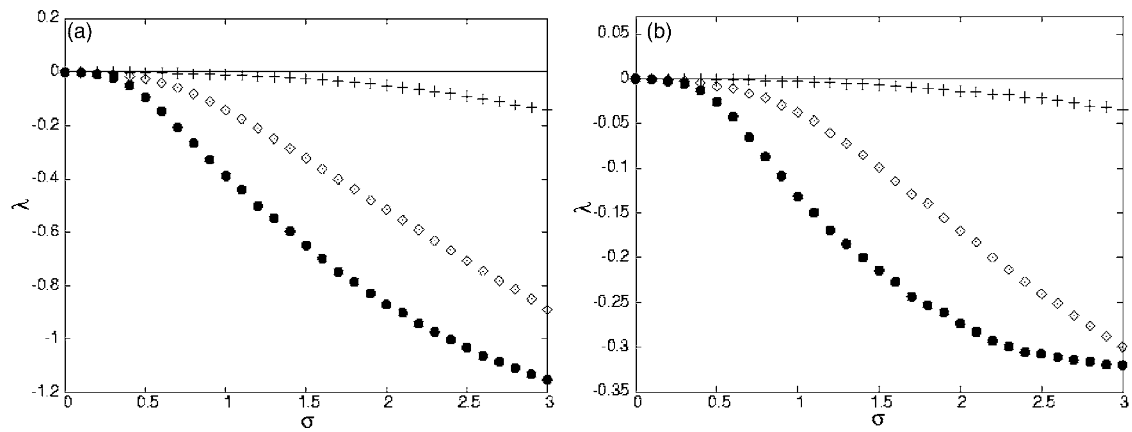


FIG. 2. LCLE λ vs noise amplitude $\sigma$ for limit cycle oscillator with $\gamma=0.5$, (●) 5 (◇), and 50 (+): (a) Stuart-Landau oscillator with $c=1$ and (b) van der Pol oscillator with $c=1$.

$D>0$ in Eq. (10). This implies that the CNIS occurs for any $\sigma>0$ when $\gamma$ is fixed because the amplitude of the OU noise is given by $\sigma=\sqrt{D\gamma}$. Let $\sigma_c$ be the synchronization threshold such that the CNIS occurs for $\sigma>\sigma_c$. We consider $\sigma_c$ as a function of $\gamma$. Since $\gamma$ is arbitrary in the above argument, we may conclude that $\sigma_c(\gamma)=0$ for any $\gamma>0$ in any limit cycle oscillators having smooth functions $\boldsymbol{f}$. The time correlation of the noise has no effect on the synchronization threshold in limit cycle oscillators. This result is consistent with that of Ref. [5], where an approximate expression for λ is derived for Gaussian noise with an arbitrary color.

We carried out numerical experiments for two examples to confirm the above analytical result. One is the noise-driven Stuart-Landau oscillator

$$\dot{\psi}=(1+ic)\psi-|\psi|^2\psi+\varepsilon(t), \tag{15}$$

where $\psi\in\mathbf{C}$ and $c$ is a constant. The other is the noise-driven van der Pol oscillator

$$\ddot{x}+c(x^2-1)\dot{x}+x=\varepsilon(t), \tag{16}$$

where $x\in\mathbf{R}$ and $c$ is a constant. Figures 2(a) and 2(b) show the LCLE as a function of $\sigma$ for three different values of $\gamma$. In both examples, it is clearly seen that for all the values of $\gamma$ the LCLE is zero at $\sigma=0$ and becomes negative as $\sigma$ becomes positive. This agrees with the analytical result.

## IV. CNIS IN CHAOTIC SYSTEMS

We proceed to the case of chaotic systems described by the equation of the form

$$\dot{\boldsymbol{x}}=\boldsymbol{f}(\boldsymbol{x},\boldsymbol{x}_\tau)+\boldsymbol{c}\varepsilon(t), \tag{17}$$

where $\boldsymbol{x}\in\mathbf{R}^n$, $\boldsymbol{x}_\tau(t)\equiv\boldsymbol{x}(t-\tau)$ with the time delay constant $\tau$, $\boldsymbol{f}$ is a smooth function, $\boldsymbol{c}\in\mathbf{R}^n$ is a constant vector, and $\varepsilon(t)$ is the OU noise defined by Eq. (1). Equation (17) includes the case of time-delay systems. The noise-free system $\dot{\boldsymbol{x}}=\boldsymbol{f}(\boldsymbol{x},\boldsymbol{x}_\tau)$ is assumed to be chaotic.

We start with an example of chaotic systems with no time delay. In this case, the LCLE is defined by Eq. (11). We consider the Lorenz system

$$\dot{x}=a(y-x),$$

$$\dot{y}=bx-y-xz+\varepsilon(t),$$

$$\dot{z}=-cz+xy, \tag{18}$$

with the parameters $a=10$, $b=28$, and $c=8/3$. It has been demonstrated in [7] that this Lorenz system exhibits the CNIS when driven by Gaussian white noise. This system exhibits the CNIS also in the OU noise case: For each fixed $\gamma$, the CNIS is observed when $\sigma$ is larger than a threshold $\sigma_c$. The threshold $\sigma_c(\gamma)$, which was determined from the LCLE, is plotted against $\gamma$ in Fig. 3(a).

Next, we consider the one-dimensional time-delay chaotic systems of the form

$$\beta^{-1}\dot{x}(t)=-cx(t)+\alpha\varphi(x(t-1))+\varepsilon(t), \tag{19}$$

where $x\in\mathbf{R}$, $\varphi$ is a smooth nonlinear function, and $\alpha$, $\beta$, and $c$ are positive constants. The time delay constant is set as $\tau=1$ without loss of generality because $\tau$ can always be made unity by rescaling the variables and the parameters. We have found sufficient conditions for the CNIS to occur when the system is driven by Gaussian white noise. These conditions reveal that system (19) exhibits the CNIS for a rather wide class of the function $\varphi$ when driven by Gaussian white noise. For example, the conditions are satisfied when $\varphi$ has the property $\lim_{x\to\pm\infty}\varphi'(x)=0$, or when $\varphi$ is a continuous and periodic function. Examples of time-delay systems used in later numerical experiments are included in one of these two cases. Thus it is expected that they also exhibit the CNIS for the OU noise. Derivation of the sufficient conditions will be presented elsewhere.

The linearized equation is obtained as

$$\dot{\Delta}(t)=\beta[-c\Delta(t)+\alpha\varphi'(x(t-1))\Delta(t-1)], \tag{20}$$

where $\Delta(t)$ represents the variation. We define the LCLE for the system (19) as follows [8]:

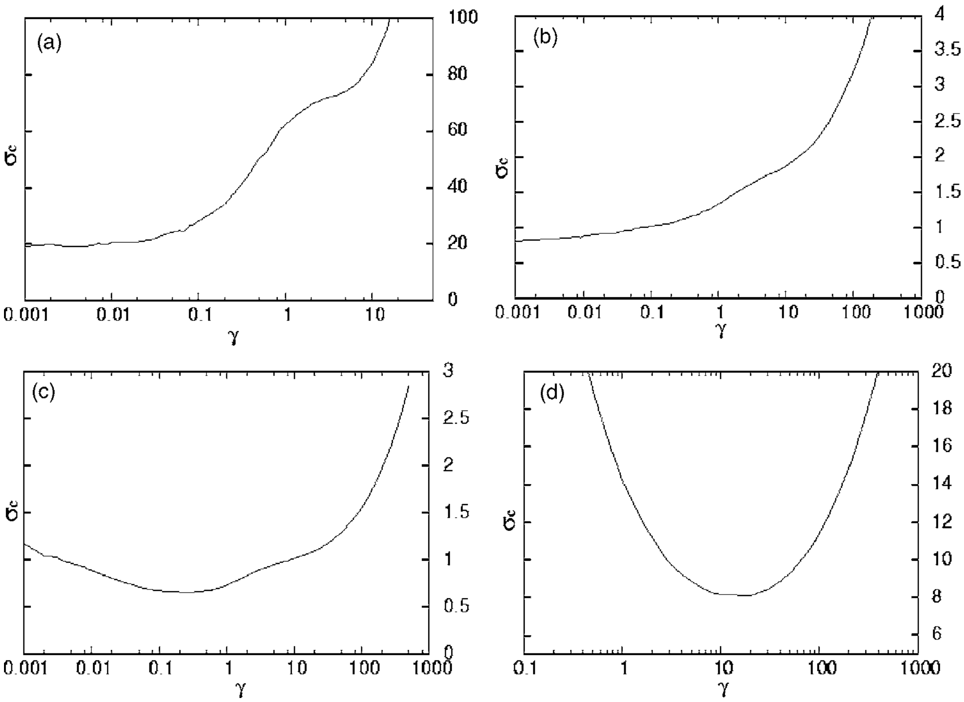


FIG. 3. Threshold noise amplitude $\sigma_c$ vs noise correlation decay rate $\gamma$ for (a) Lorenz system with $(a,b,c)=(10,28,8/3)$, (b) MG system with $p=4$ and $(\alpha,\beta,c)=(5,20,1)$, (c) MG system with $p=10$ and $(\alpha,\beta,c)=(2,20,1)$, and (d) Ikeda system with $(\alpha,\beta,c)=(4,5,1)$.

$$\lambda = \lim_{t\to\infty}\frac{1}{t}\ln\frac{\left\{\int_{t-1}^{t}\Delta^2(s)ds\right\}^{1/2}}{\left\{\int_{-1}^{0}\Delta^2(s)ds\right\}^{1/2}}. \tag{21}$$

We employ two examples of the nonlinear function $\varphi$: The first one is the Mackey-Glass (MG) system

$$\varphi(x) = \frac{x}{1+x^p}, \tag{22}$$

where $p$ is a positive integer, and the other is the Ikeda system

$$\varphi(x) = \sin x. \tag{23}$$

Figures 3(b)–3(d) show $\sigma_c(\gamma)$ for the MG systems with the parameters $p=4$ and $(\alpha,\beta,c)=(5,20,1)$, the MG systems with $p=10$ and $(\alpha,\beta,c)=(2,20,1)$, and the Ikeda system with $(\alpha,\beta,c)=(4,5,1)$, respectively.

Figure 3 shows that $\sigma_c(\gamma)$ strongly depends on $\gamma$. It should be emphasized that in this sense the time correlation effect exists in chaotic systems, in contrast to the case of limit cycle oscillators. In all of Figs. 3(a)–3(d), qualitatively similar behaviors of $\sigma_c(\gamma)$ are observed for large $\gamma$: $\sigma_c(\gamma)$ diverges with increasing $\gamma$. Consequently, a common feature is that in each example there is a finite optimal value in $\gamma$, at which $\sigma_c$ is minimized: i.e., the CNIS can be achieved by the smallest noise amplitude when the noise correlation decay rate $\gamma$ is chosen as the optimal one. Figures 3(a) and 3(b) indicate that $\sigma_c(\gamma)$ decreases as $\gamma$ approaches zero and the minimum point is at $\gamma=0$ for the Lorenz system and the MG system with $p=4$ while, in Figs. 3(c) and 3(d), the minimum points are not zero but they are located at $\gamma\simeq 0.2$ for the MG system with $p=10$ and $\gamma\simeq 15$ for the Ikeda system. The finite optimal $\gamma$ can take either zero or a positive value, depending on the particular system.

We give an explanation for the behavior of $\sigma_c(\gamma)$ in Fig. 3. Suppose that (i) the noise-free system is chaotic, (ii) $\lambda=\lambda(\gamma,\sigma)$ is a continuous function of $\gamma$ and $\sigma$, and (iii) there exists $\sigma_c(\gamma)$ for each $\gamma$ such that $\lambda<0$ for $\sigma>\sigma_c$, which is determined by solving the equation $\lambda(\gamma,\sigma_c)=0$ with respect to $\sigma_c$. Note that in terms of the parameter $D$, the assumption (iii) means that there exists a critical noise intensity $D_c(\gamma)$ for each $\gamma$ such that $\lambda<0$ for $D>D_c$: $D_c$ can be related with $\sigma_c$ as $D_c=\sigma_c(\gamma)^2/\gamma$.

Consider the limit $\gamma\to\infty$ with fixed $D$, in which the OU noise $\varepsilon$ becomes the Gaussian white noise, i.e., $\varepsilon(t)=\eta(t)$, and Eq. (17) reads

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{x}_\tau) + \boldsymbol{c}\sqrt{2D}\xi(t), \tag{24}$$

where $\xi(t)$ represents the normal Gaussian white noise, i.e., $\langle\xi(t)\rangle=0$ and $\langle\xi(t)\xi(s)\rangle=\delta(t-s)$. The LCLE $\lambda$ is determined for each $D$ from Eq. (24) in the white noise limit. Then the critical noise intensity $D_c(\infty)$ can be obtained and it equals $\lim_{\gamma\to\infty}D_c(\gamma)$. An important point here is that $D_c(\infty)$ cannot be zero but has a strictly positive finite value or diverges, because $\lambda>0$ in the case of $D=0$ from the assumption (i) and thus some finite noise intensity is necessary to make $\lambda$ negative even if $D_c(\infty)$ is finite. For large but finite $\gamma\gg 1$, the critical noise amplitude can be approximated by $\sigma_c \simeq \sqrt{D_c(\infty)\gamma}$ when $D_c(\infty)$ has a finite value, implying that $\sigma_c(\gamma)$ has to diverge in the limit $\gamma\to\infty$. Of course, $\sigma_c(\gamma)$ diverges when $D_c(\infty)$ diverges. This explains the numerically observed behavior of $\sigma_c$ in the large $\gamma$ region.

Since $\sigma_c(\gamma)$ is a continuous function of $\gamma$ and diverges in the limit $\gamma\to\infty$, the $\sigma_c(\gamma)$ has to have the minimum at a certain finite value of $\gamma$, which is either zero or positive as shown in Fig. 3: The case of $\lim_{\gamma\to\infty}\sigma_c(\gamma)=\inf_{0\leqslant\gamma}\sigma_c(\gamma)$

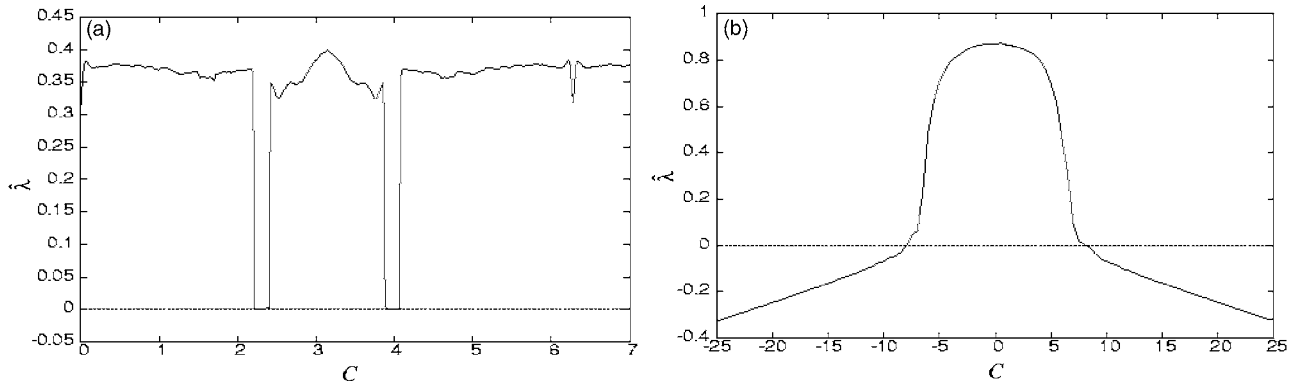


FIG. 4. LCLE for constant input for (a) Ikeda system and (b) Lorenz system. Parameters are the same as Fig. 3.

never occurs. The minimum of $\sigma_c(\gamma)$ never equals zero because $\lambda > 0$ in the noise-free case and thus a finite noise amplitude is necessary to make $\lambda$ negative.

The above argument clearly also applies to other chaotic systems, for instance, the systems described by partial differential equations. Therefore, based on the numerical results in Fig. 3 and the above argument, we may conclude that the divergence of $\sigma_c$ in the limit $\gamma \to \infty$ and the existence of a finite optimal value of the noise correlation decay rate $\gamma$ are universal characteristics of the CNIS in any chaotic systems with or without time delays. The optimal $\gamma$ is either zero or nonzero, depending on the particular system.

Finally, we discuss a condition for a system to have a nonzero value of the optimal $\gamma$, relating the LCLE in small $\gamma$ region with that for a constant input. In the case of $\gamma \simeq 0$, $\varepsilon(t)$ varies very slowly in time. Therefore, in Eq. (17), the noise term $\varepsilon(t)$ can be approximated by a constant during an appropriately chosen time interval of the length $T$. The constant is the average of $\varepsilon(t)$ over the interval. The interval length is chosen in such a way that it will be long enough for calculating the LCLE and the variation of $\varepsilon(t)$ over $T$ will be small. Let $C_m$ be the average of $\varepsilon(t)$ over the interval $(m-1)T \leq t < mT$. For small $\gamma$, the LCLE may be approximated by

$$\lambda = \lim_{N\to\infty} \frac{1}{N} \sum_{m=1}^{N} \hat{\lambda}(C_m), \tag{25}$$

where $\hat{\lambda}(C)$ is the LCLE obtained for the chaotic system subjected to the constant input, i.e., $\varepsilon(t) = C$. Equation (25) indicates that in the case of $\gamma \simeq 0$ the LCLE $\hat{\lambda}(C)$ for constant input plays an important role in determining $\lambda$. The stationary probability distribution for the values taken by $\varepsilon(t)$ is given by

$$P(\varepsilon;\sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{\varepsilon^2}{2\sigma^2}\right]. \tag{26}$$

If we replace the sum in Eq. (25) with the integral over this distribution, we can obtain the following approximation for the LCLE:

$$\lambda = \int_{-\infty}^{\infty} \hat{\lambda}(C) P(C;\sigma) dC. \tag{27}$$

If $\hat{\lambda}(C) \geq 0$ for any constant input $C \in \mathbf{R}$ and $\hat{\lambda}(C)$ is not identically zero, it follows from Eq. (27) that $\lim_{\gamma\to 0} \lambda(\gamma,\sigma) > 0$ for any $\sigma$. Then the threshold $\sigma_c(\gamma)$, which is determined from the equation $\lambda(\gamma,\sigma_c) = 0$, has to diverge as $\gamma$ approaches zero. In this case, the system has a positive nonzero optimal $\gamma$. Therefore, one sufficient condition for the system to have a nonzero optimal $\gamma$ is given by $\hat{\lambda}(C) \geq 0$ and $\hat{\lambda}(C) \neq 0$.

We show $\hat{\lambda}(C)$ for the Ikeda and Lorenz systems in Figs. 4(a) and 4(b), respectively. Because of the periodicity of the function $\varphi(x)$, $\hat{\lambda}(C)$ also has the period $2\pi$ for the Ikeda system. Figure 4(a) shows that $\hat{\lambda}(C) \geq 0$ over the whole interval of $C$, indicating that the above sufficient condition is satisfied. Based on this result, we can understand the numerical observation that $\sigma_c(\gamma)$ diverges as $\gamma$ approaches zero and a nonzero optimal $\gamma$ appears in the Ikeda system. In contrast, the sufficient condition is not satisfied in the Lorenz and MG systems. As an example, in Fig. 4(b), $\hat{\lambda}(C)$ becomes negative in the region $|C| > 7$ for the Lorenz system. Since the contribution of negative $\hat{\lambda}(C)$ becomes large in Eq. (27) as $\sigma$ increases, $\lim_{\gamma\to 0} \lambda(\gamma,\sigma)$ becomes negative for $\sigma$ larger than a certain critical value. This explains the fact that $\sigma_c(\gamma)$ does not diverge but converges to the finite critical value as $\gamma$ approaches zero.

## V. PARAMETER MISMATCH CASE

In the real world, the two driven systems are not identical but there are necessarily differences in their parameters. Therefore, it is also important to examine how the existence of parameter mismatch influences the quality of CNIS. Consider the two common-noise-driven systems of the form

$$\dot{\boldsymbol{x}}_i = \boldsymbol{f}(\boldsymbol{x}_i, \boldsymbol{x}_{i,\tau}; c_i) + \boldsymbol{c}\varepsilon(t), \quad i = 1,2, \tag{28}$$

where $\boldsymbol{x}_i \in \boldsymbol{R}^n$, $\boldsymbol{x}_{i,\tau}(t) \equiv \boldsymbol{x}_i(t-\tau)$, $c_i$ is the parameter, $\boldsymbol{f}$ is a smooth function, $\boldsymbol{c} \in \boldsymbol{R}^n$ is a constant vector, and $\varepsilon(t)$ is the OU noise defined by Eq. (1). The identical synchronization $\boldsymbol{x}_1 = \boldsymbol{x}_2$ cannot be achieved in the parameter mismatch case

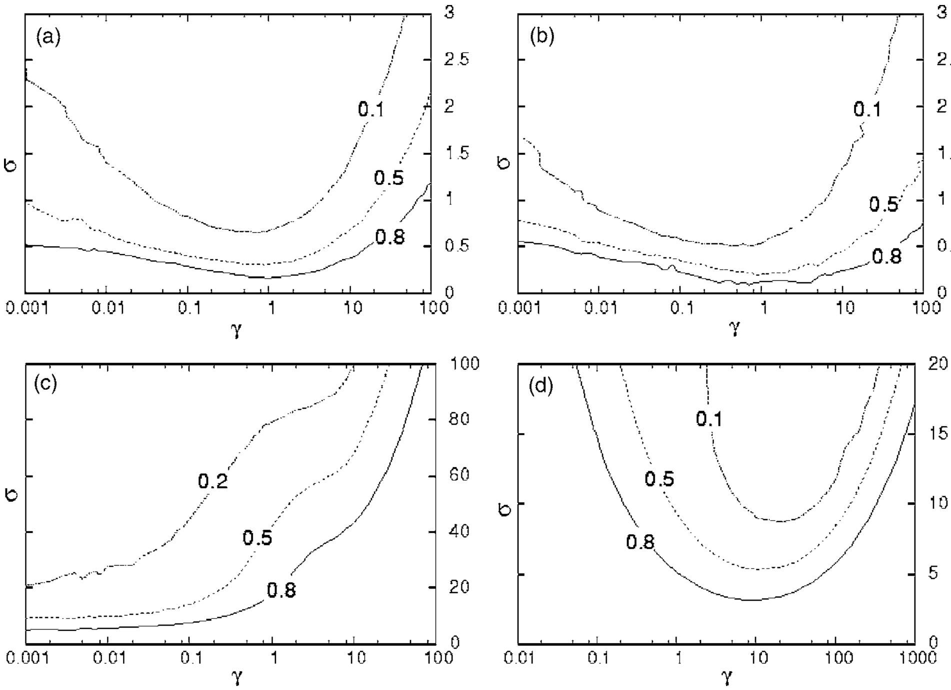


FIG. 5. Contour plot of normalized synchronization error $\bar{e}(\gamma,\sigma)$ in parameter mismatch case $c_1=c$ and $c_2=1.01\times c$ for (a) Stuart-Landau oscillator with $c=1$, (b) van der Pol oscillator with $c=1$, (c) Lorenz system with $(a,b,c)=(10,28,8/3)$, and (d) Ikeda system with $(\alpha,\beta,c)=(4,5,1)$.

$c_1\neq c_2$. We consider the case that $c_1$ and $c_2$ are slightly different and examine how the synchronization error depends on the noise amplitude and the correlation decay rate.

We carried out numerical experiments for the Stuart-Landau oscillator (15), the van der Pol oscillator (16), the Lorenz system (18), and the Ikeda system (19) with Eq. (23). In each numerical experiment, a slight parameter mismatch is introduced in the parameter $c$ and the other parameters are the same between the two driven systems: $c_1=c$ and $c_2=1.01\times c$, where $c=8/3$ for the Lorenz system and $c=1$ for the other three systems. We define the average synchronization error $e(\gamma,\sigma)$ by

$$e(\gamma,\sigma)=\lim_{T\to\infty}\frac{1}{T}\int_0^T\|\boldsymbol{x}_2(t;\gamma,\sigma)-\boldsymbol{x}_1(t;\gamma,\sigma)\|dt, \qquad (29)$$

where $\boldsymbol{x}_i(t;\gamma,\sigma)$ represents the solution of Eq. (28) in the case that the correlation decay rate is $\gamma$ and the noise amplitude is $\sigma$. In order to measure the synchronization quality, we use the normalized synchronization error $\bar{e}(\gamma,\sigma)$ defined by

$$\bar{e}(\gamma,\sigma)=\frac{e(\gamma,\sigma)}{e(\gamma,0)}. \qquad (30)$$

Contour plots of $\bar{e}(\gamma,\sigma)$ are shown in Figs. 5(a)–5(d). It should be noted that in each figure there exists a region of small $\bar{e}$ when $\sigma$ is large. This indicates that high quality CNIS is possible even in the case of small parameter mismatch. Figures 5(a) and 5(b) show the results for the Stuart-Landau oscillator and the van der Pol oscillator, respectively. In these limit cycle system cases, for any fixed $\sigma$, the error $\bar{e}$ varies with $\gamma$ and has a minimum at an intermediate value of $\gamma$. In this sense, the quality of CNIS depends on the correlation decay rate $\gamma$ for constant $\sigma$. The results for the Lorenz and Ikeda systems are shown in Figs. 5(c) and 5(d), respectively. In the case of chaotic systems, it has been observed in their time evolution that the synchronization error becomes large intermittently while it remains small in the other periods of time. The $\gamma$ dependence of the quality of CNIS is observed also in these chaotic system cases. The error $\bar{e}$ has a minimum at $\gamma=0$ for the Lorenz system and at $\gamma\simeq 20$ for the Ikeda system when $\sigma$ is constant. These values of $\gamma$ for the minimum error coincide with those of $\gamma$, at which the synchronization threshold $\sigma_c(\gamma)$ is minimized in the case of no parameter mismatch.

## VI. CONCLUSIONS

In conclusion, we studied the synchronization induced by a common OU noise input in limit cycle and chaotic systems, focusing on effects of the time correlation of noise. We showed that the threshold noise amplitude $\sigma_c(\gamma)$ exhibits completely different behaviors in limit cycle and chaotic systems: $\sigma_c(\gamma)=0$ holds for any limit cycle system defined by smooth vector field; in contrast, $\sigma_c(\gamma)$ diverges in the limit $\gamma\to\infty$ and has the nonzero minimum value at a finite optimal $\gamma$, which is either zero or positive, for any chaotic system exhibiting the CNIS. The time correlation effect is observed only in chaotic systems. Each of these two behaviors of $\sigma_c(\gamma)$ is universal for limit cycle systems and chaotic systems, respectively. A sufficient condition for the nonzero optimal $\gamma$ was also given for chaotic systems. It remains an open question how the synchronization threshold depends on the time correlation of noise for other noiselike inputs different from the OU noise. Finally, we studied the quality of CNIS in the case that there is a parameter mismatch between the two driven systems. It was demonstrated that high quality CNIS is possible even in the case of small parameter mismatch.

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### APPENDIX: NUMERICAL INTEGRATION SCHEME

We describe the numerical scheme, which we used for integrating differential or delay-differential equations driven by the OU noise. Consider the set of equations

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{x}_\tau) + \boldsymbol{c}\varepsilon, \qquad \text{(A1)}$$

$$\dot{\varepsilon} = -\gamma\varepsilon + \gamma\eta(t), \qquad \text{(A2)}$$

where $\boldsymbol{x} \in \mathbf{R}^n$, $\varepsilon \in \mathbf{R}$, and $\boldsymbol{c} \in \mathbf{R}^n$ is a constant vector. The Gaussian white noise $\eta$ has the properties $\langle \eta(t) \rangle = 0$ and $\langle \eta(t)\eta(s) \rangle = 2D\delta(t-s)$.

If we use the notations $\boldsymbol{X} = (\boldsymbol{x}, \varepsilon)^T$, $\boldsymbol{F}(\boldsymbol{X},\boldsymbol{X}_\tau) = [\boldsymbol{f}(\boldsymbol{x},\boldsymbol{x}_\tau) + \boldsymbol{c}\varepsilon, -\gamma\varepsilon]^T$, and $\boldsymbol{C} = (\boldsymbol{0}, \gamma)^T$, where $T$ means the transposed vector, then Eqs. (A1) and (A2) are rewritten in the form

$$\dot{\boldsymbol{X}} = \boldsymbol{F}(\boldsymbol{X},\boldsymbol{X}_\tau) + \boldsymbol{C}\eta(t). \qquad \text{(A3)}$$

Hereafter, we describe the scheme for the stochastic equation of this form. We divide Eq. (A3) into two parts

$$\dot{\boldsymbol{X}} = \boldsymbol{F}(\boldsymbol{X},\boldsymbol{X}_\tau), \qquad \text{(A4)}$$

and

$$\dot{\boldsymbol{X}} = \boldsymbol{C}\eta(t). \qquad \text{(A5)}$$

Let $(\Delta\boldsymbol{X})_d$ and $(\Delta\boldsymbol{X})_s$ be the variations in $\boldsymbol{X}$ for a small time step $\Delta t$, which are determined from the deterministic part (A4) and the stochastic part (A5), respectively. For sufficiently small $\Delta t$, $\boldsymbol{X}(t+\Delta t)$ can be approximated by

$$\boldsymbol{X}(t+\Delta t) \simeq \boldsymbol{X}(t) + (\Delta\boldsymbol{X})_d + (\Delta\boldsymbol{X})_s. \qquad \text{(A6)}$$

The variation $(\Delta\boldsymbol{X})_d$ is computed by using the fourth-order Runge-Kutta method to Eq. (A4). The stochastic equation (A5) can be solved analytically: The probability distribution for the $i$th component of $(\Delta\boldsymbol{X})_s$ is given by the Gaussian distribution such that the average is zero and the standard deviation is $C_i\sqrt{2D\Delta t}$, where $C_i$ is the $i$th component of $\boldsymbol{C}$. Therefore, according to the Box-Muller algorithm, $(\Delta\boldsymbol{X})_s$ can be generated from two random numbers $r_1$ and $r_2$, which are uniformly distributed on the interval $[0,1]$, as follows:

$$(\Delta\boldsymbol{X})_s = \boldsymbol{C}\sqrt{-4D\Delta t \ln r_1}\cos(2\pi r_2). \qquad \text{(A7)}$$

The numerical integration of Eq. (A3) is performed by iterating the formula (A6).

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